CALCULATION OF SCATTERING IN RADIATIVE HEAT-EXCHANGE PROCESSES

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The Bubnov-Galerkin method is used to solve the problem of the passage of isotropic radiation through a scattering layer with a spherical scattering indicatrix and the problem of radiation from a layer exhibiting constant temperature.

1. In problems of radiative exchange in dust media, scattering plays an important role. The complexity of calculating the scattering involves the difficulties related to calculating the scattering indicatrix (curve), the attenuation factor, and the scattering coefficient for the particles of the medium, and the difficulties involved in solving the equations of transport with an arbitrary scattering curve. Even for spherical curves the existing analytical formulas, based on the Schwarzschild-Schuster and Eddington approximations, are not sufficiently accurate. Methods have now been developed which make it possible to solve the problems of transport theory with virtually any degree of accuracy [1, 2]. However, to obtain a numerical solution with unlimited accuracy requires considerable work on the part of the programmers and much machine time, whereas the actual accuracy of these calculations, applied to specific conditions, frequently proves to be low because of the coarse approximation of the scattering curve or because of marked geometric simplification. Until now it has therefore been important to find methods of calculation that were valid for a specific interval of optical thicknesses and for a specific interval of particle diameters.

2. As demonstrated in [3], if the particle radius a is sufficiently large

$$\rho = \frac{2\pi a}{\lambda} > 10,$$

the attenuation factor, the scattering coefficient, and the scattering curve, calculated by the methods of geometric optics, with consideration of classical Fraunhofer diffraction, are in good agreement with the rigorous Mie theory [4], if we average over some interval of particle diameters. The portion of the radiation scattered by the particles as a result of classical diffraction is equal to 1 in units of πa^2, and amounts to half the attenuation factor for the particle, calculated by the method of geometric optics, with consideration of diffraction. This portion is markedly extended forward. Virtually all the scattered light is concentrated within a cone exhibiting a half-angle of $1/\rho$ [4]. If in passing through the layer a single quantum of radiation is scattered n times on the average, its deflection from its original direction as a result of diffraction from the medium particles involves an angle of order n/ρ, if n is small. However, in terms of order of magnitude n is equal to the optical layer thickness τ_0 and therefore if $\tau_0/\rho \gg 1$, the diffracted light may be held to coincide in direction with the incident light. If the absorption index of the particle substance satisfies the inequality $\varkappa > \lambda/\pi a$ (where λ is the radiation wavelength), the light refracted by the particles can be neglected. The scattering curve for the reflected light, for large reflecting particles, with an increase in the complex refractive index, tends to the spherical, while the coefficient of reflection can be calculated on the basis of geometric optics [5]. Thus, for large absorbing particles and for optical thicknesses not too large, the problem reduces to the solution of the equations of transport from a spherical scattering curve.

3. Let us examine the transport equation for a flat layer for the case in which intensity is independent of the azimuth angle and the scattering curve is spherical:

$$\mu \frac{\partial I}{\partial \tau} + I = \frac{r}{2} \int_{-1}^{1} I(\tau, \mu')\, d\mu' + j(\tau) \qquad (1)$$

with the boundary conditions

$$I(0, \mu > 0) = 0, \quad I(\tau_0, \mu < 0) = 0. \qquad (2)$$

Equation (1) with boundary conditions (2) can be written in integral form [1]:

$$B(\tau) = \frac{r}{2} \int_0^{\tau_0} E_1|\tau - \tau'|\, B(\tau')\, d\tau' + j(\tau), \qquad (3)$$

where $B(\tau)$ is the right-hand member of Eq. (1); E_1 is the integroexponential function of 1-st order. Subsequently, E_n will denote the integroexponential function of n-th order. We will solve (3) with the δ-like source function $j = \delta(\tau - \tau_1)$ by the Bubnov-Galerkin method [6]. The solution of the equation for the Green function

$$G(\tau, \tau_1) = \frac{r}{2} \int_0^{\tau_0} E_1|\tau - \tau'|\, G(\tau', \tau)\, d\tau' + \delta(\tau - \tau_1)$$

is sought in the form of the series

$$G = \sum_{m=0}^{N} a'_m \tau^m. \qquad (4)$$

If we limit ourselves to only a single term in the expansion, we will obtain

$$a'_0 = \frac{1}{(1-r)\tau_0 + \frac{r}{2}\varepsilon_0(\tau_0)}, \qquad (5)$$

where $\varepsilon_0(\tau_0) = 1 - 2E_3(\tau_0)$ is the dimensionless radiation energy of a uniformly heated, nonscattering layer of optical thickness τ_0.

Table 1

Energy of isotropic radiation passing through a purely scattering layer

τ_0	Results taken from [8]	After Adrianov and Polyak	Second Galerkin approximation	Second Ivon approximation
0	1.0000	1.0000	1.0000	1.0000
0.1	0.9103	0.9159	0.9154	0.9127
0.3	0.7960	0.7941	0.7936	0.7878
0.5	0.7062	0.7051	0.7044	0.6989
0.8	0.6057	0.6057	0.6048	0.6013
1.0	0.5523	0.5543	0.5534	0.5510
2.0	0.3909	0.3905	0.3901	0.3896
3.0	0.3016	0.3019	0.3006	
10.0	0.1209	0.1168	0.1170	

If in expansion (4) we consider two terms, for the coefficients a_0' and a_1' we obtain the expressions

$$a_0' = \left\{\left[\frac{1-r}{3}\tau_0^3 + r\varphi(\tau_0)\right] - \frac{\tau_0\tau_1}{2}\left[(1-r)\tau_0 + \frac{r}{2}\varepsilon_0(\tau_0)\right]\right\} \times \left\{\left[(1-r)\tau_0 + \frac{r}{2}\varepsilon_0(\tau_0)\right]\left\{\frac{1-r}{3}\tau_0^3 + r\varphi(\tau_0) - \frac{\tau_0^2}{4}\left[(1-r)\tau_0 + \frac{r}{2}\varepsilon_0(\tau_0)\right]\right\}\right\}^{-1}, \quad (6')$$

$$a_1' = (\tau_0/2 - \tau_1) \Big/ \left\{\left[\frac{1-r}{3}\tau_0^3 + \tau\varphi(\tau_0)\right] - \frac{\tau_0^2}{4}\left[(1-r)\tau_0 + \frac{r}{2}\varepsilon_0(\tau_0)\right]\right\}. \quad (6)$$

Here $\varphi(\tau_0)$ denotes the function

$$\varphi(\tau_0) = \frac{\tau_0^2}{4} - \left[\frac{1}{4} - \tau_0 E_4(\tau_0) - E_5(\tau_0)\right].$$

Knowing the Green function $G(\tau, \tau')$, we can solve the problem with any source function in the form

$$B(\tau) = \int_0^{\tau_0} j(\tau_1) G(\tau, \tau_1)\, d\tau_1 = a_0 + a_1\tau + \ldots \quad (7)$$

Here it is natural that the second approximation of the method will depend on whether the function $B(\tau)$ is well described by the expression $a_0 + a_1\tau$. The radiation intensity is expressed in terms of the function $B(\tau)$:

$$I(\tau, \mu > 0) = \int_0^{\tau} \frac{1}{\mu} \exp\left[-\frac{\tau - \tau'}{\mu}\right] B(\tau')\, d\tau',$$

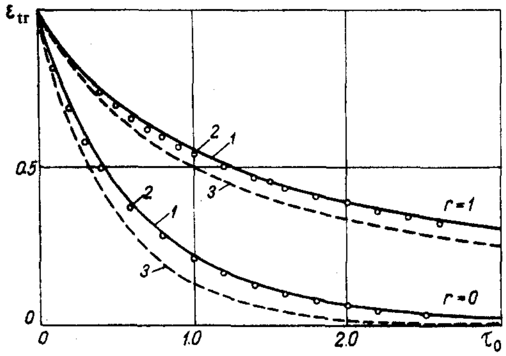
$$I(\tau, \mu < 0) = \int_{\tau_0}^{\tau} \frac{1}{\mu} \exp\left[-\frac{\tau - \tau'}{\mu}\right] B(\tau')\, d\tau', \quad (8)$$

while the energy passing beyond the boundary $\tau = \tau_0$ is expressed by

$$\varepsilon = \int_0^{\tau_0} E_2(\tau_0 - \tau') B(\tau')\, d\tau'. \quad (9)$$

4. Let us examine the passage of isotropic radiation of unit energy through the layer.

Equation (1) with boundary conditions (2) for $j = rE_2(\tau)$ describes the scattered radiation. We will not carry out the calculations in second approximation of the method because they are analogous to the calculations of Kuznetsov [7] who sought the solution for functions B also in the form $a_0 + a_1\tau$ for $r = 1$. The problem of the passage of isotropic radiation through a purely scattering layer has been solved by many authors; this is because the problem of equilibrium heat exchange between a layer without scattering (or with a spherical scattering curve) and black walls [8] or, as



Energy of isotropic radiation through layer as function of optical layer thickness (r = 1 is the spherical indicatrix): 1) exact solution; 2) Ivon's second approximation (circles); 3) Schwarzschild–Schuster approximation.

Table 2

Dimensionless energy of layer radiation and energy of isotropic radiation as functions of the optical layer thickness τ_0

Dimensionless energy of layer radiation (r = 0.5)				Energy of isotropic radiation passing through the layer (r = 0.5)			
τ_0	Second Ivon approximation	Schwarzschild-Schuster approximation	First Galerkin approximation	τ_0	First Galerkin approximation	Schwarzschild-Schuster approximation	Second Ivon approximation
0.1	0.0929	0.094	0.0911	0.1	0.8665	0.861	0.871
0.2	0.1733	0.181	0.170	0.2	0.7589	0.740	0.767
0.5	0.3593	0.398	0.357	0.5	0.5300	0.482	0.543
0.8	0.4897	0.537	0.492	0.8	0.3813	0.320	0.398
1.0	0.5561	0.603	0.562	1.0	0.3083	0.237	0.328
1.5	0.6734	0.707	0.684	1.5	0.1830	0.112	0.214
2.0	0.7445	0.768	0.762	2.0	0.1091	0.058	0.150

can be demonstrated, walls that are gray but isotropically reflecting, reduces to the first-cited problem.

Hottel [8] derived exact results for the energy passing through the layer, solving the problem on a computer. Table 1 shows a comparison of the results for the magnitude of the passing energy, calculated in second approximation of the Bubnov-Galerkin method, with the exact calculation of Hottel, as well as with the Adrianov and Polyak calculations [9]. The comparison shows excellent agreement for the second approximation of the Bubnov-Galerkin method.

For thin layers we can limit ourselves to only a single term in the expansion of function B and use the first approximation. For the radiation energy passing through the layer we have the formula

$$\varepsilon_{tr} = 2E_3(\tau_0) + \frac{r}{4}\,\frac{\varepsilon_0^2(\tau_0)}{(1-r)\tau_0 + \frac{r}{2}\varepsilon_0(\tau_0)}. \qquad (10)$$

When r = 1 this formula transforms to the familiar [9]

$$\varepsilon_{tr} = \frac{1+2E_3(\tau)}{2}.$$

In first approximation of the Galerkin method it is also not difficult to derive a formula for the dimensionless energy of radiation for a uniformly heated layer

$$\varepsilon_{rad} = \frac{(1-r)\tau_0\varepsilon_0(\tau_0)}{(1-r)\tau_0 + \frac{r}{2}\varepsilon_0(\tau_0)}. \qquad (11)$$

Of course, when $r \neq 0$ this formula is not valid for thick layers.

5. Table 2 shows a comparison of calculations, from (10) and (11), with the solution of the Schwarzschild-Schuster method for r = 0.5, as well as with the second approximation of the Ivon method [2]. We conducted the calculation in accordance with the Ivon method on an M-3M computer.

Comparison (see figure) of the exact solution (r = = 1 and r = 0) for the energy of isotropic radiation passing through the layer with the second Ivon approximation, as well as with the Schwarzschild-Schuster method, which is a first approximation of the Ivon method for the spherical scattering curve, shows that the second Ivon approximation is fairly exact. We see from Table 2 that for small optical thicknesses the calculation according to (10) and (11) is more exact than the calculation according to the Schwarzschild-Schuster method. If the layer is not thin, the calculation of the radiation that passes through the layer can be carried out in second approximation of the Galerkin method. To evaluate the accuracy and to refine the solution we can employ the method of iterations, taking only a single iteration.

6. Thus, if the first Galerkin approximation is taken as the zero approximation in the method of iterations for the problem of layer radiation, for the energy of radiation from the layer we derive the following formula:

$$\varepsilon_{rad} = \varepsilon_0(\tau_0)(1-r) + \frac{r(1-r)\tau_0}{(1-r)\tau_0 + \frac{r}{2}\varepsilon_0(\tau_0)} \times$$
$$\times \left[\varepsilon_0(\tau_0) - G'_{22}(\tau_0) - G_{22}(\tau_0)\right], \qquad (12)$$

where

$$G'_{22}(\tau_0) = \int_0^{\tau_0} E_2(\tau_0 - \tau')E_2(\tau')\,d\tau',$$

$$G_{22}(\tau_0) = \int_0^{\tau_0} E_2^2(\tau')\,d\tau'.$$

The last integral can be calculated approximately by using the approximate representation of the function $E_2(\tau)$:

$$E_2(\tau) \approx \frac{1}{2}\left[\exp[-(3+\sqrt{3})\tau] + \right.$$
$$\left. + \exp[-(3-\sqrt{3})\tau]\right]. \qquad (13)$$

Formula (13) is derived in the following manner. Since

$$E_2(\tau) = \int_0^1 \exp\left[-\frac{\tau}{\mu}\right] d\mu,$$

to find E_2 we have to find the approximate expression for the function $\exp[-\tau/\mu]$, which is a solution of the equation $\mu(dI/d\tau) + I = 0$ with the boundary condition $I(0, \mu > 0) = 1$. If we solve this equation in second approximation of the Ivon method, we obtain

$$\exp\left[-\frac{\tau}{\mu}\right] \approx \frac{1}{2\sqrt{3}}\left\{(3+\sqrt{3})\left[1 - \frac{6\mu}{3+\sqrt{3}}\right] \times \right.$$
$$\times \exp[-\tau(3+\sqrt{3})] +$$
$$\left. + (3-\sqrt{3})\left[\frac{6\mu}{3-\sqrt{3}} - 1\right]\exp[-(3-\sqrt{3})\tau]\right\}.$$

From this we obtain (13). It then is not difficult to find

$$G'_{22}(\tau) = \frac{\tau}{2}E_2(\tau) + \frac{1}{2\sqrt{3}}E_2(\tau) -$$
$$- \frac{1}{2\sqrt{3}}\exp[-(3+\sqrt{3})\tau], \qquad (14)$$

$$G_{22}(\tau) = \frac{1}{4}\left\{\frac{1-\exp[-2(3+\sqrt{3})\tau]}{2(3+\sqrt{3})} + \right.$$
$$\left. + \frac{1-\exp[-6\tau]}{3} + \frac{1-\exp[-2(3-\sqrt{3})\tau]}{2(3-\sqrt{3})}\right\}. \qquad (15)$$

NOTATION

a is the particle radius; λ is the radiation wavelength; ρ is the quantity $2\pi a/\lambda$; τ is the optical thickness; $\varkappa$ is the index of absorption of substance of the particles; I is the radiation intensity; r is the ratio of the dispersion factor to attenuation factor; μ is the cosine of the angle between τ-axis and the ray direction; E_n

is the integroexponential function of n-th order; δ is the Dirac delta function; j is the source function; G is the Green's function; $\varepsilon_0(\tau_0)$ is the dimensionless energy of radiation of a uniformly heated nonscattering layer with optical thickness τ_0; ε_{tr} is the dimensionless energy of isotropic radiation passed through a layer having optical thickness τ_0; ε_{rad} is the dimensionless energy of radiation of a flat layer with optical thickness τ_0 having a constant temperature.

REFERENCES

1. S. Chandrasekhar, The Transfer of Radiant Energy [Russian translation], IL, 1953.
2. B. Davison, The Theory of Neutron Transfer [Russian translation], Atomizdat, 1960.
3. J. R. Hodkinson and I. J. Greenleaves, Opt. Soc. Amer., **53**, 577, 1963.
4. K. S. Shifrin, The Scattering of Light in a Murky Medium [in Russian], Moscow-Leningrad, 1951.
5. A. P. Prishivalko and Yu. I. Chekalinskaya, collection: Spectroscopy of Light Scattering Media [in Russian], Izd. AN BSSR, 1963.
6. S. G. Mikhlin, Variational Methods in Mathematical Physics [in Russian], Gostekhizdat, 1957.
7. E. S. Kuznetsov, Transactions of the Institute for Theoretical Geophysics, 1 [in Russian], Izd. AN SSSR, 1946.
8. H. C. Hottel, Int. J. Heat Mass Transf., **5**, 82, 1962.
9. V. N. Adrianov and G. L. Polyak, Int. J. Heat Mass Transf., **6**, 355, 1963.

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